Black Hole Astrophysics
Chapters 6.5.2. & 6.6.2.2 & 7.1~7.3

All figures extracted from online sources of from the textbook.

Flowchart

Fluids in Special Relativity → Conservation laws and the equations of motion

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The Electromagnetic Stress Tensor

What's wrong with Newtonian Gravity?

↓

What does matter do to spacetime? → What does it mean to be curved?

↓

How can we more quantitatively describe curvature? → Gravity & Spacetime

Gravity & Spacetime → Meaning of gauge, How to calculate the metric

Meaning of gauge → GR without sources

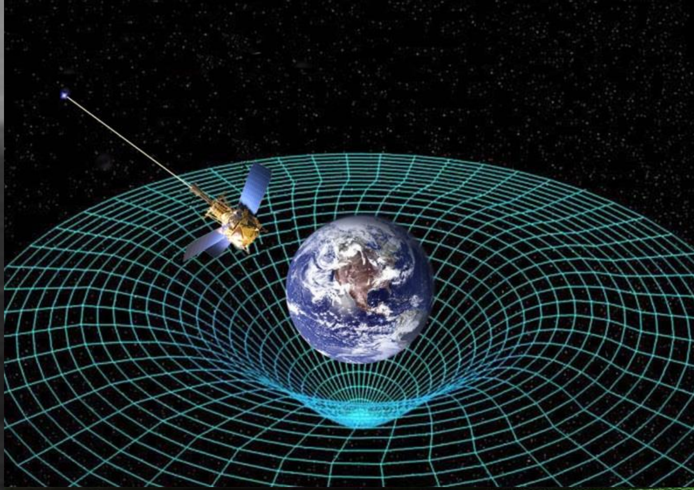
How to calculate the metric → The Schwarchild metric

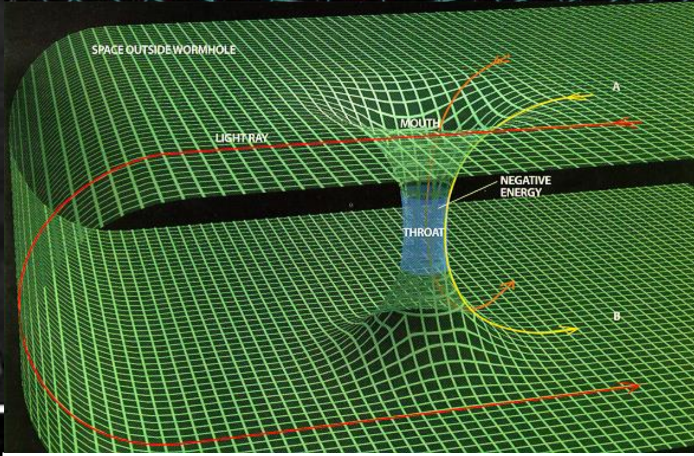
GR without sources → The Schwarchild metric

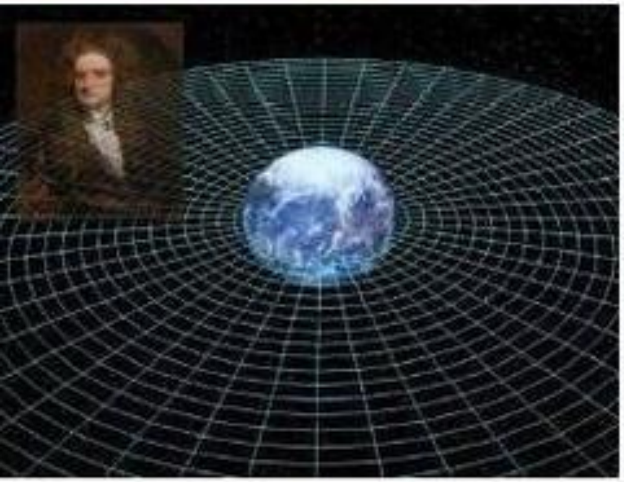
Fluids in Special Relativity

The equations of motion

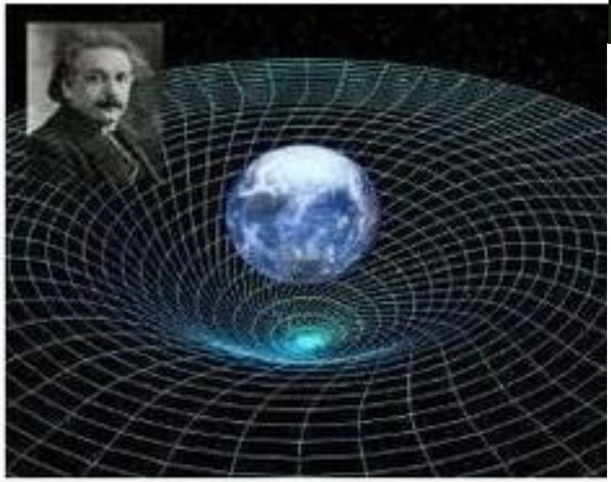
The Electromagnetic Stress Tensor



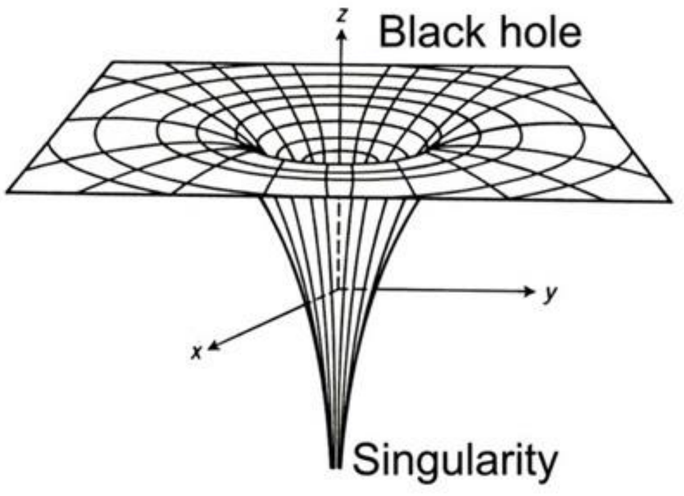




Newton's fixed space



Einstein's flexible space-time



Overview

The special theory of relativity can describe an enormous amount of physics. It can handle forces due to fluid or gas pressure, flow speeds up to the speed of light, electric and magnetic forces, and even viscous forces. One aspect that is completely missing, however, is gravity. And the gravitational force cannot be added easily: it cannot be written in a stress-energy tensor form for use in equation (6.121). This is true even in Newtonian mechanics, where it is included as an add-on "body force". In order to include gravity in the theory of relativity, Einstein reasoned that gravity must be a pseudo-force, arising not from another stress-energy component, but from the gradient operator itself (∇) in equation (6.121). In other words, because gravity occurs when matter is present, somehow matter must cause four-dimensional space to be curved, rather than flat. This curvature then gives rise to additional terms in the equations of motion that we interpret as the force of gravity. The addition of curvature to the spacetime metric, plus the realization that energy is mass, and therefore can partake in the generation of the gravitational field, led to a complete and consistent theory of gravity that we now know as Einstein's general theory of relativity [308, 309]. Constructing a theory of gravity that was consistent with four-dimensional spacetime, however, was a monumental task that took Einstein nearly ten years to fully work out. And it has taken the century following that for the rest of us to determine its implications.[1]

What's wrong with Newtonian Gravity?

Recall the basic assumptions of Special Relativity:
1. Physics is the same in any inertial frame of reference
2. Speed of light is the same for all observers

How about Newtonian Gravity?

1. The effects of gravity propagates instantaneously

↓

Violates the speed limit of special relativity.

2. $\nabla^2\phi = 4\pi G\rho$ only contains ρ term, it has different values in different frames!

↓

ρ can't be a scalar quantity but actually part of the stress energy tensor as we have seen.

↓

We need a proper tensor theory of Gravity that reduces to produce Newtonian Gravity in limiting cases.

↓

By extension of $\nabla^2\phi = 4\pi G\rho$, we would expect the tensor form to look like $G^{\alpha\beta} = KGT^{\alpha\beta}$. Where $T^{\alpha\beta}$ contains ρ as one of its terms.

So, our goal today is …

1. To find $G^{\alpha\beta}$ that will produce the Einstein equation $G^{\alpha\beta} = KGT^{\alpha\beta}$ and find the proportionality constant K

2. Apply it to find the metric of a Schwarzschild Black Hole

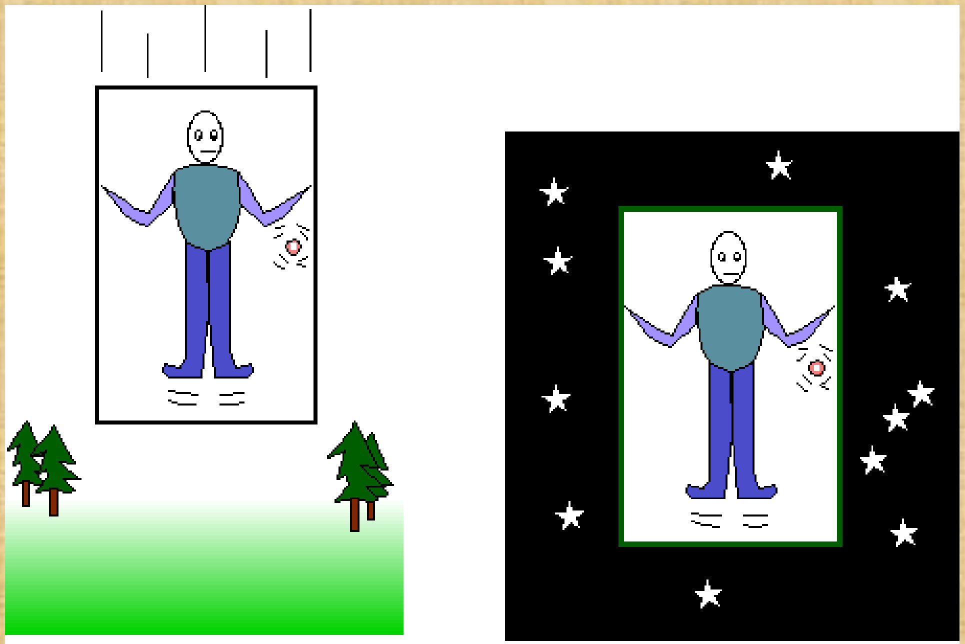
Let’s cut the cable

Previously, we have discussed properties of spacetime using the Minkowski metric. There, we never took gravity into account.

But as far as we knew, for any observer in an inertial frame, he feels spacetime behaving as the Minkowski metric tells it to.

Consider a freely falling elevator, people, as well as anything in it will feel as if there were no gravity. Basically a similar concept as the “imaginary force” used in classical mechanics to describe accelerating frames.

Einstein took an extension of this idea to say that the freely falling frame is the actual inertial one! Therefore people in it see spacetime behaving as the Minkowski metric says!



Things falling freely in a gravity field all accelerate by the same amount, so they move the same way as if they were in a region of zero gravity — ``weightlessness''!

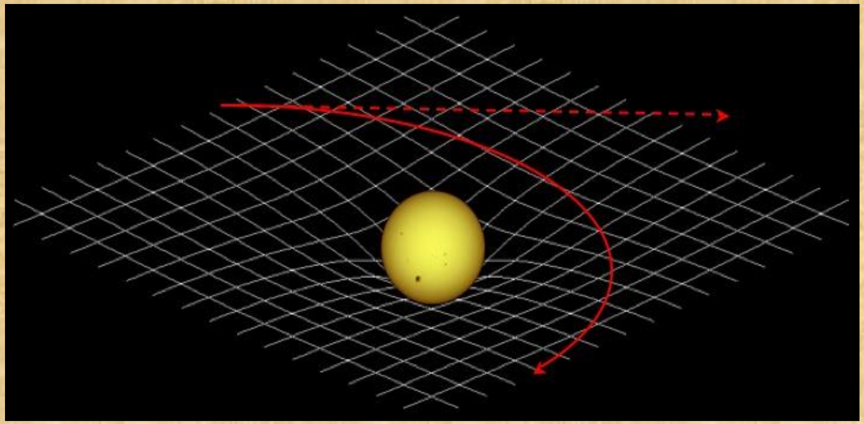
What does all this mean for spacetime?

Now, <u>*assuming*</u> freely falling frames are inertial, this means that different observers placed in a non-uniform gravity field must be in different accelerating frames relative to each other.

An observer freely falling sees others accelerating with respect to him!

When there is gravity around, it becomes impossible to create a inertial frame in which no one is accelerating! Thus, the metric now is a function of position! Different observers at different locations are in different accelerating frames thus measures things differently!

Sorry, I don't have a very simple way of connecting these two concepts directly.



We call this type of spacetime as being curved! Spacetime is being bent by the existence of mass-energy!

What does it mean to be curved?

The basic idea: Two initially parallel lines don't remain parallel after extending in the original direction.

Simple method: Since initially parallel lines remain parallel in Euclidian space, if we can find a global transform between the metric of interest and the Euclidian one, then it must be flat.

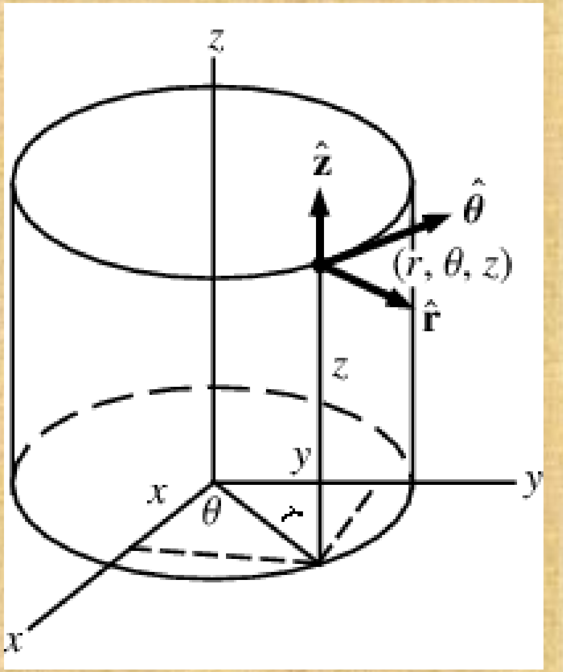
To determine if there is a global transform between the two, review the section in 6.2.4

For the surface of a cylinder,

$$\mathrm{dh}^2 = R^2\mathrm{d}\phi^2 + \mathrm{dz}^2 = (d[\mathrm{R}\phi])^2 + \mathrm{dz}^2$$

Then, redefining $d[\mathrm{R}\phi] \rightarrow \mathrm{dx}$; $\mathrm{dz} \rightarrow \mathrm{dy}$, we retain $\mathrm{dh}^2 = \mathrm{dx}^2 + \mathrm{dy}^2$

Therefore, the surface of a cylinder is flat!



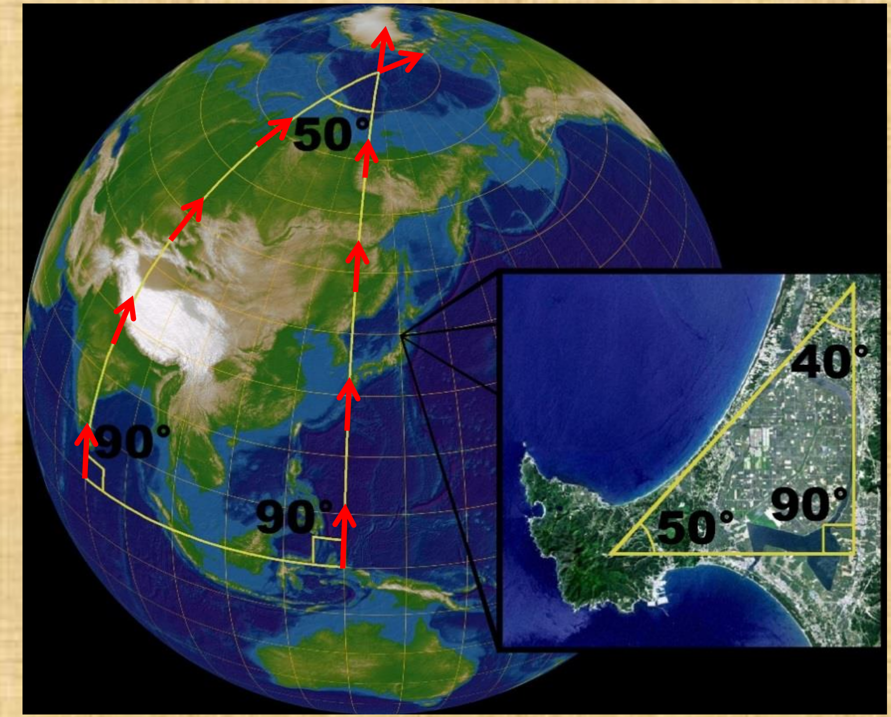
What does it mean to be curved?

For the surface of a sphere,

$$dh^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

We find it is impossible to find a global transform to retain $dh^2 = dx^2 + dy^2$

Therefore, the surface of a sphere is curved!



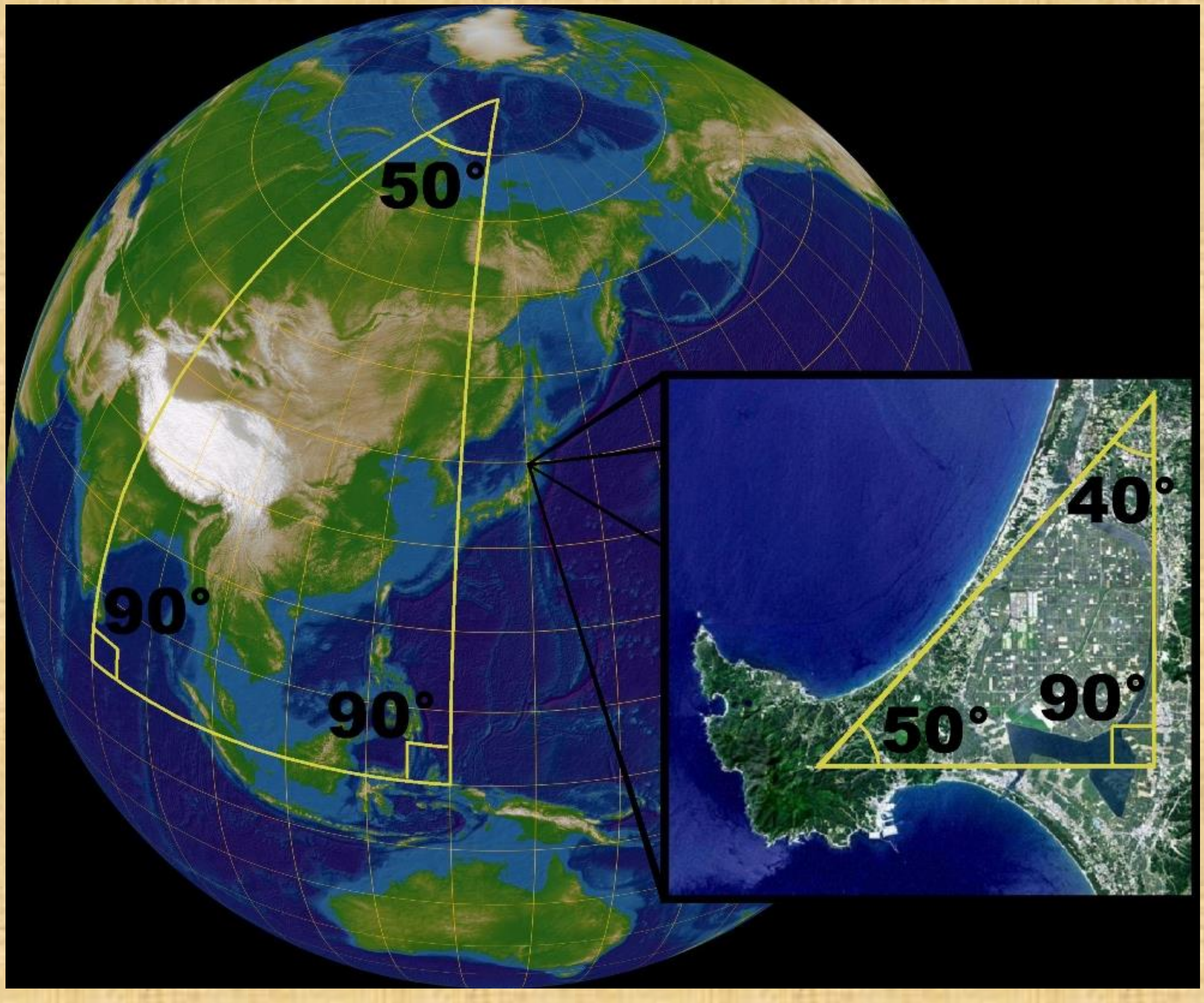
Another way is to consider the figure on the left:

At the equator, there are two parallel lines (since both are at right angles to the equator).

However, extending them and one finds that they meet at the pole! They are no longer parallel!

A connection to Gravity

Before we continue, one interesting thing to note here is that although the Earth as a whole is a curved a surface. But we never draw town maps as if they were on a curved surface.



What this means is that locally, a small patch of the sphere looks more or less flat!

Recall that a few slides ago, we said that spacetime is curved by matter, but as we have also shown, locally, we can still find some frame that looks follows the Minkowski metric. Thus, gravity also has the similar property of being locally flat but globally curved!

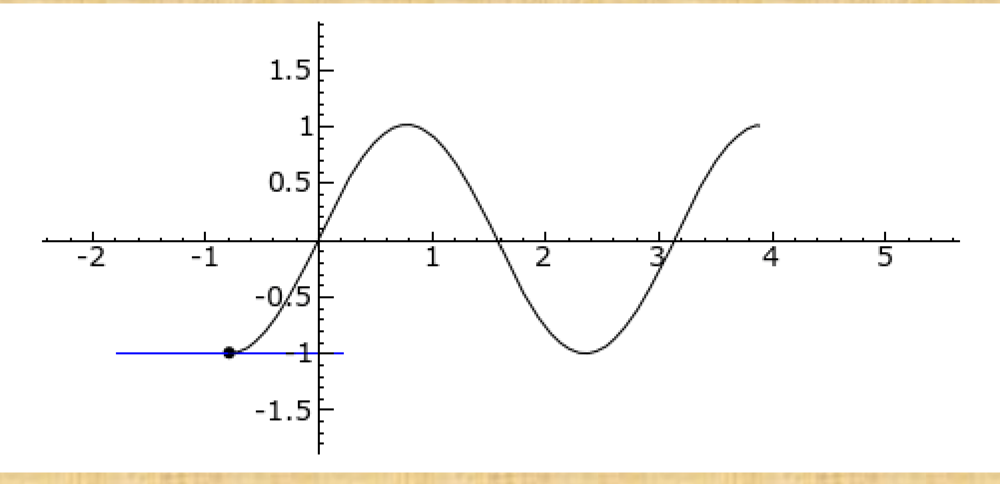
How do we test curvature more precisely?

Although illustrative, but for higher dimensions it wouldn't be trivial to "see" if two initially parallel lines remain parallel or even try to do global transformations on them. Therefore, to treat higher dimension spaces, we use the method introduced by Georg Friedrich Bernhard Riemann in the mid 1800s – to compute the Riemann tensor.

Recall that in Calculus, we learned that for a 1D function f(x),

The first derivative $\frac{\mathrm{df}(x)}{\mathrm{dx}}$ gives us the slope of the function at some point,

whereas the second derivative $\frac{\mathrm{df}(x)^2}{\mathrm{dx}^2}$ gives us the curvature (concave/convex).



A hell lot of terms!

To compute the Riemann tensor also requires taking second derivatives. We need to *consider every possible second derivative* on *each term of the metric*.

For N dimensional space, there are $N \times N$ possible second derivatives and also $N \times N$ terms in the metric.

This amounts to N^4 terms for N dimensional spaces. (256 for 4D!)

For example, in 4D spacetime, the Minkowski metric has $4 \times 4 = 16$ terms.

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And all possible second derivatives are as follows (also 16):

$$\frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \underline{\frac{\partial^2}{\partial t \partial x}, \frac{\partial^2}{\partial x \partial t}}, \underline{\frac{\partial^2}{\partial t \partial y}, \frac{\partial^2}{\partial y \partial t}}, \underline{\frac{\partial^2}{\partial t \partial z}, \frac{\partial^2}{\partial z \partial t}}, \underline{\frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial y \partial x}}, \underline{\frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial z \partial x}}, \underline{\frac{\partial^2}{\partial y \partial z}, \frac{\partial^2}{\partial z \partial y}}$$

However, as we can already see, there must be some symmetries in the Riemann tensor that will allow us to get rid of lots of terms.

The Riemann Tensor

Without going into detail, the Riemann tensor is defined as:

$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + g^{\mu\nu}\left(\Gamma_{\mu\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma_{\mu\alpha\gamma}\Gamma_{\nu\beta\delta}\right)$$

We can see that it is composed of the difference of two terms

Where the $\Gamma_{\alpha\beta\gamma}$ terms are related to the metric as:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

BAD SLIDE

They are called the Christoffel symbols.

Now, we can see that the Riemann tensor is related to the metric through the second derivative.

Meaning of the Christoffel symbols

This shows explicitly that the derivative of $\vec{V}$ is more than just the derivative of its components V^α. Now, since r is just one coordinate, we can generalize the above equation to

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\,\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}, \qquad (5$$

where, now, x^β can be either r or θ for $\beta = 1$ or 2.

BAD SLIDE

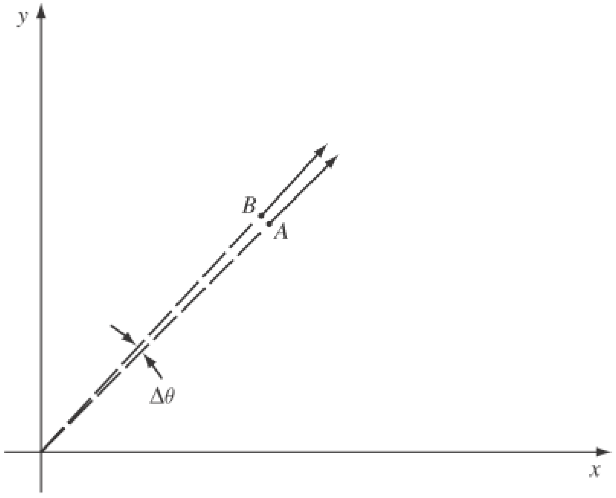
A general vector V has components (V^r, V^θ) on the polar basis. Its derivative, by analog with Eq. (5.40), is

$$\begin{aligned}\frac{\partial \vec{V}}{\partial r} &= \frac{\partial}{\partial r}(V^r\,\vec{e}_r + V^\theta\,\vec{e}_\theta)\\ &= \frac{\partial V^r}{\partial r}\,\vec{e}_r + V^r\frac{\partial \vec{e}_r}{\partial r} + \frac{\partial V^\theta}{\partial r}\,\vec{e}_\theta + V^\theta\frac{\partial \vec{e}_\theta}{\partial r},\end{aligned}$$

and similarly for $\partial\vec{V}/\partial\theta$. Written in index notation, this becomes

$$\frac{\partial \vec{V}}{\partial r} = \frac{\partial}{\partial r}(V^\alpha\,\vec{e}_\alpha) = \frac{\partial V^\alpha}{\partial r}\,\vec{e}_\alpha + V^\alpha\frac{\partial \vec{e}_\alpha}{\partial r}.$$

(Here α runs of course over r and θ.)



Change in $\vec{e}_r$, when θ changes by $\Delta\theta$.

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta}\,\vec{e}_\mu$$

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\,\vec{e}_\alpha + V^\alpha\Gamma^\mu{}_{\alpha\beta}\,\vec{e}_\mu.$$

$$V^\alpha{}_{;\beta} := V^\alpha{}_{,\beta} + V^\mu\Gamma^\alpha{}_{\mu\beta}.$$

Hooray! Let's do away with the terms! Algebraic Symmetries of the Riemann Tensor

From the definition of the Riemann Tensor and Christoffel symbols,

$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + g^{\mu\nu}\left(\Gamma_{\mu\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma_{\mu\alpha\gamma}\Gamma_{\nu\beta\delta}\right) \qquad \Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

We can find that

$$R_{\alpha\beta\delta\gamma} = -R_{\alpha\beta\gamma\delta} \qquad R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta} \qquad R_{\alpha\beta\delta\gamma} = -R_{\beta\alpha\delta\gamma} \qquad R_{\alpha\alpha\gamma\delta} = R_{\alpha\beta\gamma\gamma} = 0$$

And the first Bianchi identity: $\quad R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$

Putting this all together, we find that the number of independent terms is # as follows:

$$\# = \frac{1}{8}(N^4 - 2N^3 + 3N^2 - 2N) \qquad N < 4$$

$$\# = \frac{1}{8}(N^4 - 2N^3 + 3N^2 - 2N) - \frac{N!}{4!(N-4)!} \qquad N \geq 4$$

This means that we only have 1,6,20 potentially non-zero independent terms for 2,3,4 dimensions respectively!

Differential symmetry of the Riemann Tensor

The Riemann Tensor also possesses differential symmetries, sometimes called the second Bianchi identity or simply Bianchi identity.

$$\nabla_\gamma R_{\mu\nu\alpha\beta} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\alpha R_{\mu\nu\beta\gamma} = 0$$

This identity does not decrease the number of components of the tensor that we need to calculate, but it does have profound importance for Einstein's theory of gravity and for fundamental laws of physics.

Example Riemann Tensors

Previously, we have argued that the surface of a cylinder is flat and the surface of a sphere isn’t. How do we show that using Riemann Tensors?

$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + g^{\mu\nu}\left(\Gamma_{\mu\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma_{\mu\alpha\gamma}\Gamma_{\nu\beta\delta}\right) \qquad \Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

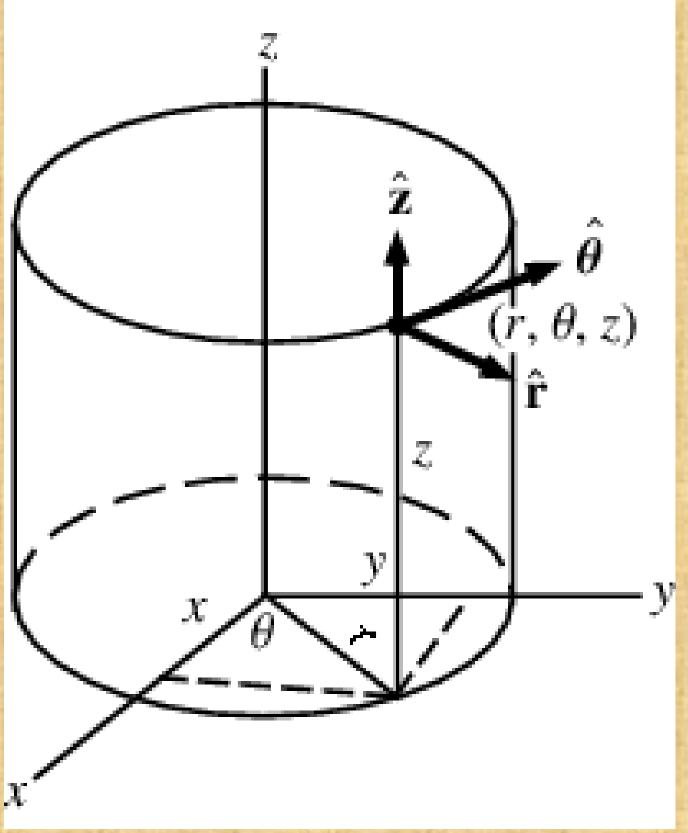
Example1.

For the surface of a cylinder, the metric is
$dh^2 = r^2 d\phi^2 + dz^2$

Since r is constant, this means that all of the Christoffel symbols $\Gamma_{\alpha\beta\gamma}$ must be zero.

Which in turn means that the Riemann tensor $R_{\alpha\beta\gamma\delta}$ is also zero.

Thus, the surface of a cylinder is flat.



Example Riemann Tensors

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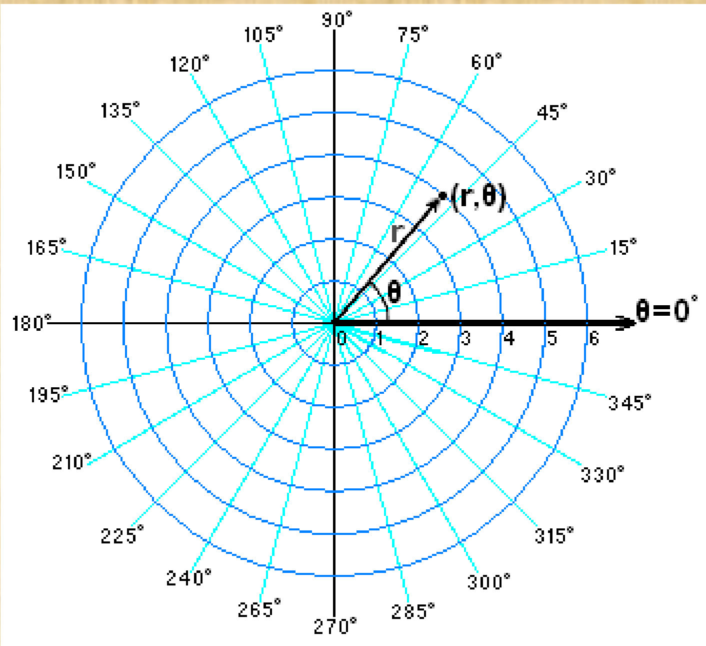
$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + g^{\mu\nu}\left(\Gamma_{\mu\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma_{\mu\alpha\gamma}\Gamma_{\nu\beta\delta}\right) \qquad \Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

Example2.

Before we go to the sphere, lets consider polar coordinates on a flat piece of paper which has the metric $\mathrm{dh}^2 = r^2 \mathrm{d}\theta^{\ 2} + \mathrm{d}r^2$



Now r isn't constant, however, for 2D problems, we only have 1 independent non-zero term.

Using $\Gamma_{212} = \Gamma_{221} = -\Gamma_{122} = r$ and $g^{11} = 1; g^{22} = 1/r^2$, we find that $R_{1212} = -1 + 1 = 0$

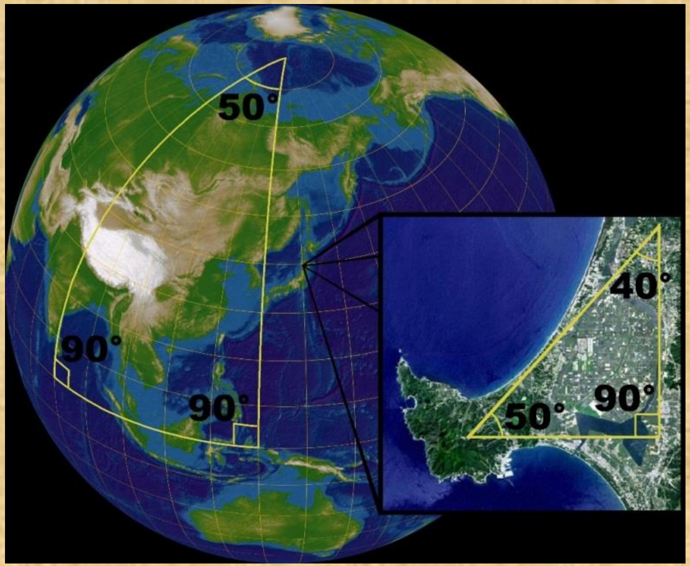
Thus, it is also flat, as expected.

Example Riemann Tensors

Previously, we have argued that the surface of a cylinder is flat and the surface of a sphere isn't. How do we show that using Riemann Tensors?

$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + g^{\mu\nu}\left(\Gamma_{\mu\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma_{\mu\alpha\gamma}\Gamma_{\nu\beta\delta}\right) \qquad \Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

Example3.



Finally, the surface of a sphere follows the metric $\mathrm{dh}^2 = r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$

This gives $\Gamma_{212} = \Gamma_{221} = -\Gamma_{122} = r^2 sin\theta cos\theta$ and $g^{11} = 1/r^2; g^{22} = 1/r^2 \sin^2\theta$

Then, $R_{1212} = r^2 \sin^2\theta$.

Initially, this looks strange: does the curvature change with position (θ) on the sphere? No, the issue is simply with how we express the Riemann tensor. If we write it as a *mixed* tensor, then the only non-zero component $\mathfrak{R}_{12}{}^{12} = 1/r^2$ is constant over the sphere.[4] So the surface of a sphere is *not* flat: the Riemann tensor has at least one non-zero component.

Now, back to gravity

Before we went into all that messy stuff of dealing with curvature, we had, based on simple analogies with Newtonian gravity argued that we had to find some equation that looked like like $G^{\alpha\beta} = KGT^{\alpha\beta}$.

As we have argued, gravity is nothing more than the manifestation of curved spacetime. Now that we have discussed all about how to describe curvature in general, we are finally in place to put things together.

However, the Riemann Tensor $R_{\alpha\beta\gamma\delta}$ is a 4^{th} rank tensor where as $G^{\alpha\beta}$ is only of 2^{nd} rank. Therefore we must somehow manipulate the Riemann Tensor into a 2^{nd} rank one.

Also, since $\nabla \cdot \overleftrightarrow{T} = 0$, we also require that $\nabla \cdot \overleftrightarrow{G} = 0$.

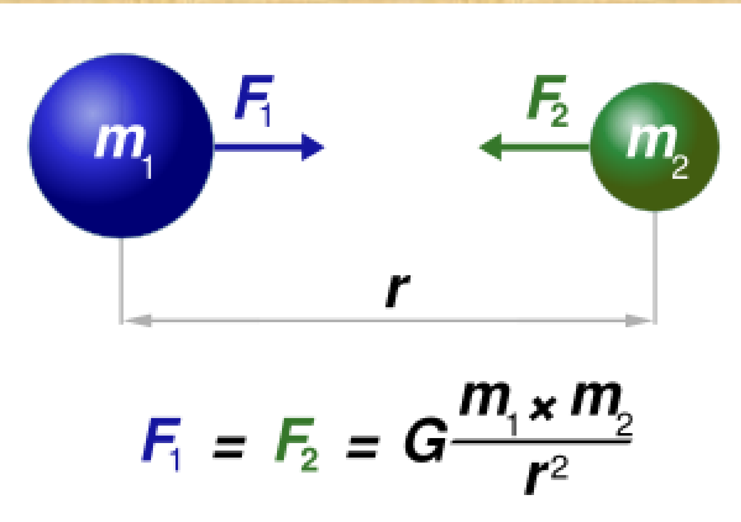
Our to do list

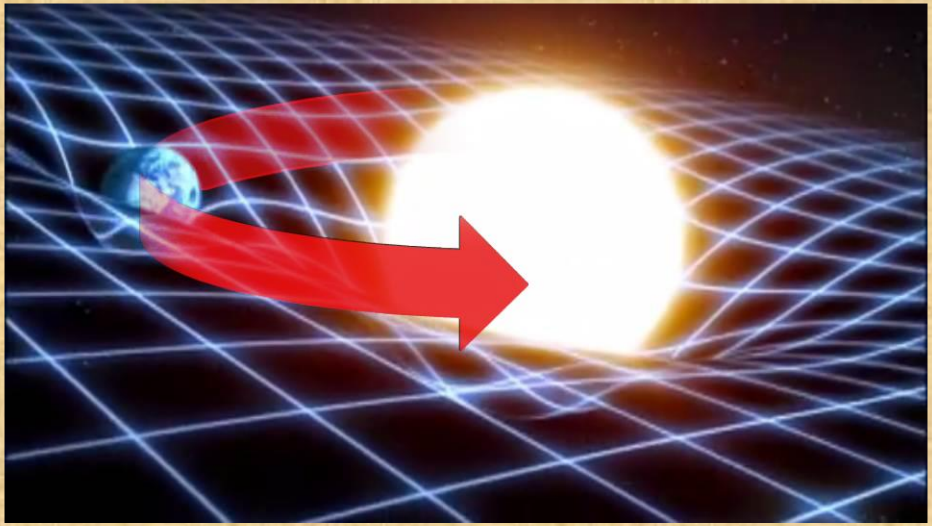
1. Find $G^{\alpha\beta}$ from the Riemann Tensor $R_{\alpha\beta\gamma\delta}$.

2. $G^{\alpha\beta}$ must satisfy $\nabla \cdot \overleftrightarrow{G} = 0$.

3. Find the proportionality constant K in $G^{\alpha\beta} = KGT^{\alpha\beta}$.

4. $G^{\alpha\beta} = KGT^{\alpha\beta}$ must reduce to $\nabla^2\phi = 4\pi G\rho$ in limiting cases.





Finding a 2^{nd} rank tensor from the Riemann Tensor

1. Find $G^{\alpha\beta}$ from the Riemann Tensor $R_{\alpha\beta\gamma\delta}$.
2. $G^{\alpha\beta}$ must satisfy $\nabla \cdot \overleftrightarrow{G} = 0$.
3. Find the proportionality constant K in $G^{\alpha\beta} = KGT^{\alpha\beta}$.
4. $G^{\alpha\beta} = KGT^{\alpha\beta}$ must reduce to $\nabla^2\phi = 4\pi G\rho$ in limiting cases.

There are many ways to extract a 2^{nd} rank tensor from a 4^{th} rank one, but only a few will inherit the properties of the Bianchi identities. (why does it have to?)

One method would be to contract the metric with the Riemann Tensor

$$R_{\alpha\beta} \equiv g^{\mu\nu} R_{\mu\alpha\nu\beta}$$

$R_{\alpha\beta}$ is called the Ricci curvature tensor.

Then, $R_{\alpha\beta}$ inherits a differential symmetry property from the Riemann tensor's Bianchi identity:

$$\nabla_\alpha\left(g^{\alpha\lambda} R_{\lambda\beta}\right) - \frac{1}{2}\nabla_\beta R = 0$$

Where

$$R \equiv g^{\beta\lambda} R_{\lambda\beta}$$

R is usually called the Ricci scalar.

Making a 2nd rank tensor that is divergence-free

~~1. Find $G^{\alpha\beta}$ from the Riemann Tensor $R_{\alpha\beta\gamma\delta}$.~~

~~2. $G^{\alpha\beta}$ must satisfy $\nabla \cdot \overleftrightarrow{G} = 0$.~~

3. Find the proportionality constant K in $G^{\alpha\beta} = KGT^{\alpha\beta}$.

4. $G^{\alpha\beta} = KGT^{\alpha\beta}$ must reduce to $\nabla^2\phi = 4\pi G\rho$ in limiting cases.

However, from $\nabla_\alpha\left(g^{\alpha\lambda}R_{\lambda\beta}\right) - \frac{1}{2}\nabla_\beta R = 0$, we can see that the Ricci tensor $R_{\alpha\beta}$ isn't divergence free.

But we can use this property to construct a divergence free tensor:

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$$

It is actually the onle 2nd rank tensor that can be derived from the Riemann tensor and has all the necessary properties that we required!

Recap—from the metric to the Einstein Tensor

~~1. Find $G^{\alpha\beta}$ from the Riemann Tensor $R_{\alpha\beta\gamma\delta}$.~~

~~2. $G^{\alpha\beta}$ must satisfy $\nabla \cdot \overleftrightarrow{G} = 0$.~~

3. Find the proportionality constant K in $G^{\alpha\beta} = KGT^{\alpha\beta}$.

4. $G^{\alpha\beta} = KGT^{\alpha\beta}$ must reduce to $\nabla^2\phi = 4\pi G\rho$ in limiting cases.

Now that we have

1. Worked out how the Riemann Tensor $R_{\alpha\beta\gamma\delta}$ related to the metric.

$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + g^{\mu\nu}\left(\Gamma_{\mu\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma_{\mu\alpha\gamma}\Gamma_{\nu\beta\delta}\right); \Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

2. Found the relation between the Riemann Tensor $R_{\alpha\beta\gamma\delta}$ and the Einstein Tensor $G^{\alpha\beta}$.

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = g_{\mu\nu} R^{\mu\alpha\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\mu\nu} R_{\mu\delta\nu\gamma}$$

We are in good position to finally tackle the proportionality constant K.

How we go about doing this is through the idea that Newtonian Gravity is actually a weak field limit of General Relativity.

The final link – the proportionality constant

~~1. Find $G^{\alpha\beta}$ from the Riemann Tensor $R_{\alpha\beta\gamma\delta}$.~~

~~2. $G^{\alpha\beta}$ must satisfy $\nabla \cdot \overleftrightarrow{G} = 0$.~~

3. Find the proportionality constant K in $G^{\alpha\beta} = KGT^{\alpha\beta}$.

4. $G^{\alpha\beta} = KGT^{\alpha\beta}$ must reduce to $\nabla^2\phi = 4\pi G\rho$ in limiting cases.

For the limit of weak gravity, we can guess that the metric should look almost Minkowskian, something like this:

$$\begin{pmatrix} -1 + h_{\mathrm{tt}} & 0 & 0 & 0 \\ 0 & 1 + h_{\mathrm{xx}} & 0 & 0 \\ 0 & 0 & 1 + h_{\mathrm{yy}} & 0 \\ 0 & 0 & 0 & 1 + h_{\mathrm{zz}} \end{pmatrix}$$

To relate them with Newtonian Gravity, all we need is $g^{\mathrm{ii}}\Gamma_{\mathrm{itt}} \approx -\frac{1}{2}\frac{\partial h_{\mathrm{tt}}}{\partial x^i}$

<u>The next bit I don't fully understand so I just cut the text out for completeness.</u>

From equation (B.5) the conservation of momentum in, *e.g.*, the x direction is

$$
\begin{aligned}
0 &= (\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{T}})^{x} \\
&= \sum_{\beta} \boldsymbol{\nabla}_{\beta}\, \mathcal{T}^{x\beta} \\
&= \sum_{\beta} \frac{\partial \mathcal{T}^{x\beta}}{\partial x^{\beta}} \;+\; \sum_{\mu\beta} g^{xx}\, \Gamma_{x\mu\beta}\, \mathcal{T}^{\mu\beta} \;+\; \sum_{\mu} \mathcal{T}^{x\mu} \sum_{\beta\lambda} g^{\beta\lambda}\, \Gamma_{\lambda\mu\beta}
\end{aligned}
$$

If $V \ll c$, then the only large component of $\mathcal{T}^{\alpha\beta}$ is

$$\mathcal{T}^{ww} = \rho\, c^2$$

so we are left with only one pseudo-force term in the full equation

$$0 = \frac{\partial(\rho \boldsymbol{V} c)}{\partial w} \;+\; \nabla \cdot (\rho\, \boldsymbol{V}\, \boldsymbol{V}) \;+\; \nabla p \;-\; \frac{1}{2} \nabla(h_{ww})\, \rho\, c^2$$

The last term in this equation of motion must be the negative of Newton's gravitational force $\rho\, \nabla\psi$, so we now can determine h_{ww} and $\mathcal{G}^{ww}$

$$h_{ww} = -2\psi/c^2 \qquad \mathcal{G}^{ww} \;=\; 2\,\nabla^2\psi/c^2$$

Therefore, if Newton's and Einstein's laws of gravitation are, respectively,

$$\nabla^2\psi \;=\; 4\pi\, G\, \rho \qquad \text{and} \qquad 2\,\nabla^2\psi/c^2 \;=\; K\, G\, \rho\, c^2$$

then, in order for the two theories to agree in the weak gravity limit, the constant of proportionality in Einstein's theory of gravity must be

$$K = \frac{8\pi}{c^4}$$

BAD SLIDE

The final Einstein field equations, then, are

$$\boxed{\boldsymbol{\mathcal{G}} \;=\; 8\pi\, \frac{G}{c^4}\, \boldsymbol{\mathcal{T}}} \tag{7.21}$$

Meaning of gauge

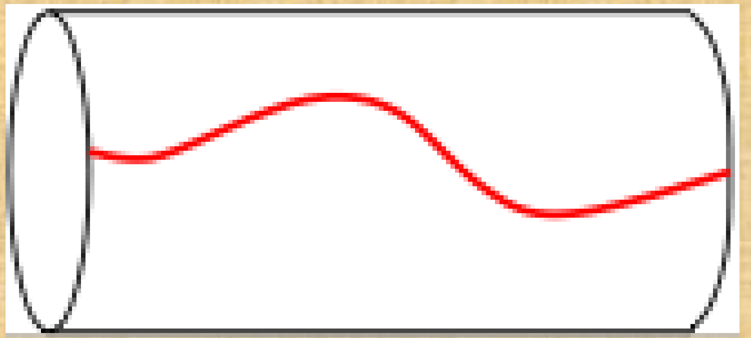
The fact that $\nabla \cdot \overleftrightarrow{G}$ tells us that the number of useful equations in $G^{\alpha\beta} = 8\pi G T^{\alpha\beta}$ reduces to only 6.

The theory thus allows for gauge freedom for us to construct 4 independent conditions on the metric that we lost as conservation laws.

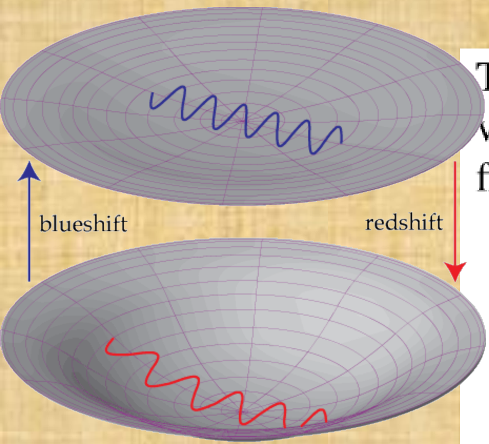
What is the interpretation of gauge in GR?

The answer is that the gauge is simply the coordinate system itself!

The gauge is the coordinate system, and the gauge transformation is the generalized Lorentz transform!



E&M in 4D curved space



The beauty and simplicity of using geometry to describe curved space is very obvious for electromagnetic theory. The same geometric equations that we derived in flat space (equations (6.112) and (6.113)) still describe the electromagnetic field

$$\nabla \cdot \mathcal{F}^{\mathrm{T}} = \frac{4\pi}{c}\mathbf{J} \tag{7.23}$$

$$\nabla \cdot \mathcal{M}^{\mathrm{T}} = 0 \tag{7.24}$$

and the equations of motion can also be expressed in the same way

$$\nabla \cdot \mathcal{T}_{\mathrm{FL}} = \frac{1}{c}\mathbf{J}^{\mathrm{T}} \cdot \mathcal{F}$$

or, by defining the electromagnetic stress-energy tensor (equation (6.121)),

$$\nabla \cdot (\mathcal{T}_{\mathrm{FL}} + \mathcal{T}_{\mathrm{EM}}) = 0$$

The only difference now is that the curvature of space and the presence of gravity are hidden in the geometry of the gradient operator ∇, which is determined by how the metric is expressed in a coordinate system. The equations of E & M are valid in any frame – flat or curved, accelerating or not – and, therefore, are generally covariant.

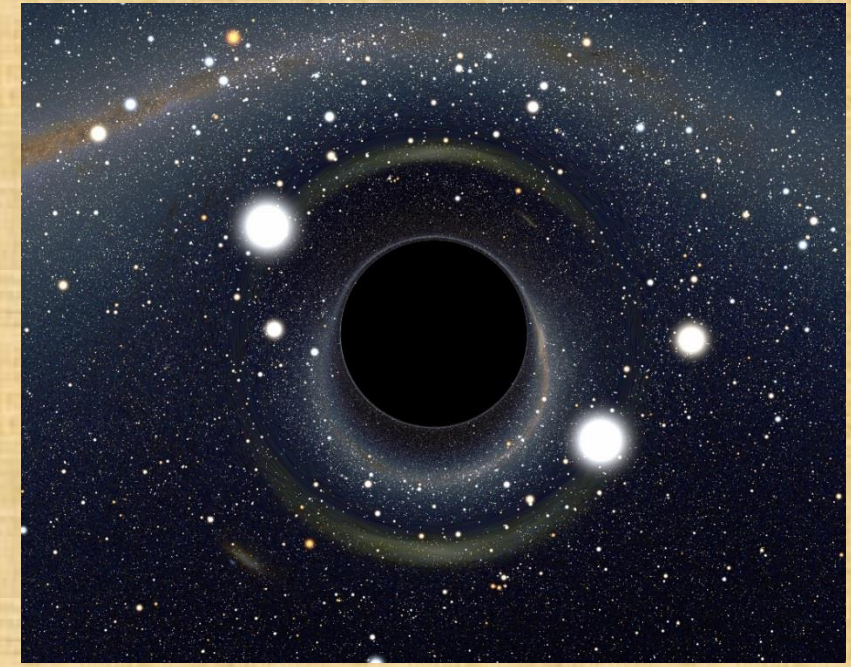
Curvature without local matter

Since $G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = g_{\mu\nu} R^{\mu\alpha\nu\beta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} g^{\mu\nu} R_{\mu\delta\nu\gamma}$,

Even if $G^{\alpha\beta} = T^{\alpha\beta} = 0$, this doesn't mean that the Riemann curvature tensor $R^{\alpha\beta\gamma\delta}$ must also be zero, i.e. not necessarily flat.

Non-zero components of $R^{\alpha\beta\gamma\delta}$ can sum to be zero when calculating $G^{\alpha\beta}$.

This actually happens quite often, as long as we are in a region without sources of curvature (i.e. the stress-energy tensor is zero there), then $G^{\alpha\beta}$ *must be zero there.*



However, we might as well be right next to a black hole where the curvature is severe! This is the situation we will be dealing with as an Example.

How to calculate a metric?

Now, let say we want to find the metric to some distribution of mass-energy as described by $T^{\alpha\beta}$.

We know that…

$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + g^{\mu\nu}\left(\Gamma_{\mu\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma_{\mu\alpha\gamma}\Gamma_{\nu\beta\delta}\right) ; \Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = g_{\mu\nu}R^{\mu\alpha\nu\beta} - \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta}g^{\mu\nu}R_{\mu\delta\nu\gamma}$$

$$G^{\alpha\beta} = 8\pi G T^{\alpha\beta}$$

So…

All we have to do is to use some assumptions to “guess” the form of the metric.

Then send the metric through the pipeline and solve the 10 equations so that $G^{\alpha\beta} = 8\pi G T^{\alpha\beta}$ is fully satisfied.

This might seem a daunting task, but lets look at the example of a lonely black hole and we’ll see that it isn’t so terrible after all.

A lonely and non-spinning BH – Assumptions to the Schwarzschild metric

-1. Any general metric

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

0. The metric is a symmetric tensor

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}$$

1. Spherically symmetric
   a. choose to use spherical coordinates $(\mathrm{t}, \mathrm{r}, \theta, \phi)$
   b. unchanged under θ,ϕ reversals $(\theta \to -\theta, \phi \to -\phi)$

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{01} & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}$$

2. Static
   a. $\frac{\partial g_{\mu\nu}}{\partial t} = 0$
   b. unchanged under t reversals $(t \to -t)$

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}$$

3. Solution is in vacuum $G^{\alpha\beta} = 8\pi G T^{\alpha\beta} = 0$

The Christoffel Symbols

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2$$

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}$$

$$g_{\alpha\beta} = \begin{pmatrix} B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\Gamma_{t\alpha\beta} = \begin{pmatrix} 0 & \frac{B'}{2} & 0 & 0 \\ \frac{B'}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_{\theta\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 0 & -r^2\sin\theta\cos\theta \end{pmatrix}$$

$$\Gamma_{r\alpha\beta} = \begin{pmatrix} \frac{-B'}{2} & 0 & 0 & 0 \\ 0 & \frac{A'}{2} & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix}$$

$$\Gamma_{\phi\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r\sin^2\theta \\ 0 & 0 & 0 & r^2\sin\theta\cos\theta \\ 0 & r\sin^2\theta & r^2\sin\theta\cos\theta & 0 \end{pmatrix}$$

Derivation of the Schwarzschild metric

$$g_{\alpha\beta} = \begin{pmatrix} B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$R_{\alpha\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^\delta} + \Gamma^\nu{}_{\alpha\delta}\Gamma_{\nu\beta\gamma} - \Gamma^\nu{}_{\alpha\gamma}\Gamma_{\nu\beta\delta}$$

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = g_{\mu\nu}R^{\mu\alpha\nu\beta} - \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta}g^{\mu\nu}R_{\mu\delta\nu\gamma}$$

$$G^{\alpha\beta} = 8\pi G T^{\alpha\beta} = 0$$

Derivation of the Schwarzschild metric

$$G^{\alpha\beta} = 8\pi G T^{\alpha\beta} = 0$$

$$4\dot{A}B^2 - 2r\ddot{B}AB + r\dot{A}\dot{B}B + r\dot{B}^2A = 0$$

$$-2r\ddot{B}AB + r\dot{A}\dot{B}B + r\dot{B}^2A - 4\dot{B}AB = 0$$

$$r\dot{A}B + 2A^2B - 2AB - r\dot{B}A = 0$$

$$\dot{A}B + \dot{B}A = 0 \rightarrow \frac{d}{\mathrm{dr}}(\mathrm{AB}) = 0 \rightarrow A(r)B(r) \equiv K$$

$$r\dot{A}\frac{K}{A} + 2KA - 2K + r\dot{A}\frac{K}{A} = 2K\left(r\frac{\dot{A}}{A} + A - 1\right) = 0 \rightarrow r\dot{A} = A(1-A)$$

$$A(r) = \frac{1}{\left(1 + \frac{1}{Sr}\right)} \qquad B(r) = K\left(1 + \frac{1}{Sr}\right)$$

$$\mathrm{ds}^2 = K\left(1 + \frac{1}{Sr}\right)\mathrm{dt}^2 + \frac{\mathrm{dr}^2}{\left(1 + \frac{1}{Sr}\right)} + r^2\mathrm{d\theta}^2 + r^2\sin^2\theta\,\mathrm{d\phi}^2$$

The coefficients

Next Week!

The Schwarzschild metric

$$\boldsymbol{g}_{\mathrm{SH}}^{\mathrm{SCH}} = \begin{pmatrix} -c^2\left(1-\frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$